ON HENRI CARTAN'S VECTORIAL MEAN-VALUE THEOREM AND ITS APPLICATIONS TO LIPSCHITZIAN OPERATORS AND GENERALIZED LEBESGUE-BOCHNER-STIELTJES INTEGRATION THEORY.

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Abstract. H. Cartan in his book on differential calculus proved a theorem generalizing a Cauchy's mean-value theorem to the case of functions taking values in a Banach space.

Cartan used this theorem in a masterful way to develop the entire theory of differential calculus and theory of differential equations in finite and infinite dimensional Banach spaces.

The author proves a generalization of this theorem to the case when the inequality involving the derivatives holds everywhere with exception of a set of Lebesgue measure zero, and the derivatives are replaced by weaker derivatives. Namely the right-sided Lipschitz derivative and lower right-sided Dini derivative, respectively.

He also presents applications of the theorem to the study of Lipschitzian operators in Banach spaces. Lipschitzian operators played pivotal role in the n-body problems of electrodynamics, as also in general n-body problem of Einstein's special theory of relativity. For references see Bogdan
http://arxiv.org/abs/0909.5240 and
http://arxiv.org/abs/0910.0538.

Using the generalization of Cartan's theorem the author proves a version of the fundamental theorem of calculus in a class of Bochner summable functions. In the process he introduces the reader to the generalized theory of Lebesgue-Bochner-Stieltjes integral and Lebesgue and Bochner spaces of summable functions as developed by Bogdanowicz.

1. INTRODUCTION

Let R and Y denote, respectively, the space of reals and a Banach space. We assume that the reader is familiar with the notion of a Banach space as defined in [1], [15], or [17]. Let $I = [a, b]$ be a closed bounded interval. Henri Cartan in his book on differential calculus [15], proved the following

Theorem: Let I be the closed interval $[a, b]$. Assume that $f: I \mapsto Y$ and $g: I \mapsto R$ are two continuous functions having right-sided derivatives $f'_r(x)$ and $g'_r(x)$ at every

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point of the open interval (a, b). If

$$\|f'_r(x)\| \leq g'_r(x) \quad \text{for all} \quad x \in (a, b),$$

then

$$\|f(b) - f(a)\| \leq g(b) - g(a).$$

Cartan used this theorem in a masterful way to develop the entire theory of *Differential Calculus* and *Theory of Differential Equations* in finite and infinite dimensional Banach spaces.

We will prove a generalization of this theorem to the case when the inequality involving the derivatives holds everywhere with exception, perhaps, of a set of Lebesgue measure zero, and the derivatives are replaced by weaker derivatives. Namely the right-sided Lipschitz derivative and lower right-sided Dini derivative, respectively.

We will also show applications of the theorem to the study of Lipschitzian operators in Banach spaces. Lipschitzian operators played pivotal role in the n-body problems of electrodynamics, as also in general n-body problem of Einstein's special theory of relativity [18]. For references see Bogdan [9]–[13].

We will show how one can use the theorem to prove a version of the Fundamental Theorem of Calculus in a class of Bochner summable functions. In the process we will introduce the reader to the generalized theory of Lebesgue-Bochner-Stieltjes integral and Lebesgue and Bochner spaces of summable functions as developed in Bogdanowicz [3]–[7].

2. Dini's one-sided derivatives

Let R denote the field of real numbers and Y a Banach space. We shall follow the notation used by Cartan [15].

Definition 2.1 (Left- and right-sided derivatives)**.** *A function $f : [a, b] \mapsto Y$ has a right-sided derivative at a point $x \in [a, b)$ if*

$$\lim_{h>0, h\to 0} \frac{1}{h}(f(x + h) - f(x))$$

exists. This limit will be denoted by $D_+f(x) = f'_r(x)$ and will be called the right-sided derivative of f at the point x.

Similarly we define the left-sided derivative $D_-f(x) = f'_l(x)$ at a point $x \in (a, b]$

$$D_-f(x) = \lim_{h<0, h\to 0} \frac{1}{h}(f(x + h) - f(x)).$$

Obviously a function f has a derivative $Df(x) = f'(x)$ at a point $x \in (a, b)$ if and only if both derivatives $f'_r(x)$ and $f'_l(x)$ exist and are equal. These derivatives represent an element from the Banach space Y. In the case of reals, Y=R, it is sometimes convenient to admit also infinite values ∞ and $-\infty$.

In some arguments it is convenient to introduce derivatives known in the literature as Dini's derivatives.

Definition 2.2 (Dini's derivatives). *Consider a real-valued function $f : (a, b) \mapsto R$. By a right-sided upper Dini derivative of f at a point $x \in (a, b)$ we shall understand the finite or infinite limit*

$$D_+^u f(x) = \limsup_{h>0, h\to 0} \frac{1}{h}(f(x+h) - f(x)).$$

Similarly we define the right-sided lower Dini derivative by

$$D_+^l f(x) = \liminf_{h>0, h\to 0} \frac{1}{h}(f(x+h) - f(x)).$$

And by analogy we define the left-sided upper derivative $D_-^u f(x)$ and the left-sided lower $D_-^l f(x)$ derivative. Clearly The function f has a right-sided derivative $D_+ f(x)$ at a point x if and only if $D_+^u f(x) = D_+^l f(x)$. Similar relation is also valid for left-sided derivatives.

3. Lipschitzian functions and Lipschitzian derivatives

Definition 3.1 (Lipschitzian functions). *Let X, Y be some Banach spaces and $I \subset X$. We shall say that a function $f : I \mapsto Y$ is Lipschitzian on the set I if there exists a constant m such that*

$$\|f(x_1) - f(x_2)\| \leq m\|x_1 - x_2\| \quad \textit{for all} \quad x_1, x_2 \in I.$$

We shall say that such a function is Lipschitzian at a point $x_0 \in I$ if there exist a constant $m < \infty$ and a positive δ such that

$$\|f(x) - f(x_0)\| \leq m\|x - x_0\| \quad \textit{for all} \quad x \in I, \|x - x_0\| < \delta.$$

When $X = R$ and $I = (a, b)$ is an open interval, we shall say that the function is Lipschitzian at a point x_0 to the right of it, if for some $m < \infty$ there exists a positive δ such that

$$\|f(x) - f(x_0)\| \leq m|x - x_0| \quad \textit{for all} \quad x \in I, x_0 < x < x_0 + \delta.$$

Similarly we define what it means that the function is Lipschitzian at a point x_0 to the the left of that point.

Definition 3.2 (Lipschitz derivatives). *Now let Y denote a Banach space and consider a function $f : (a, b) \mapsto Y$. By a right-sided Lipschitz derivative of f at a point $x \in (a, b)$ we shall understand the finite or infinite limit*

$$L_+ f(x) = \limsup_{h>0, h\to 0} \|\frac{1}{h}(f(x+h) - f(x))\|.$$

Similarly we define the left-sided Lipschitz derivative by

$$L_- f(x) = \limsup_{h<0, h\to 0} \|\frac{1}{h}(f(x+h) - f(x))\|.$$

We shall say that the function f has a Lipschitz derivative $Lf(x)$ at a point x if $L_+ f(x) = L_- f(x)$ and we denote the common value by $Lf(x)$.

Clearly if right-sided derivative $f_r'(x)$ exists then the right-sided Lipschitz derivative exists and we have the equality $\|f_r'(x)\| = L_+ f(x)$. Similar relations are valid for $f_l'(x)$ and $f'(x)$ and corresponding Lipschitz derivatives $L_- f(x)$ and $Lf(x)$.

Notice also that at a point $x \in (a, b)$ the Lipschitz derivatives

$$Lf(x),\ L_+ f(x),\ L_- f(x)$$

are finite if and only if the function is at the point x, respectively, Lipschitzian, Lipschitzian to the right, Lipschitzian to the left of the point.

4. Sets of Lebesgue measure zero

Definition 4.1 (Set of Lebesgue measure zero)**.** *A set $A \subset R$ is said to be of* **Lebesgue measure zero** *if for every $\varepsilon > 0$ there exists a countable collection of intervals $T = \{I_1, I_2, \ldots\}$ such that the set A is contained in the union of sets in T and*

$$\sum\nolimits_{I \in T} |I| \le \varepsilon,$$

where $|I|$ denotes the length of the interval I.

Clearly the empty set $\emptyset = (a, a)$, and any singleton $[b, b]$ is of Lebesgue measure zero. Moreover any countable set of points forms a set of Lebesgue measure zero. Notice also that countable union of sets of Lebesgue measure zero, is a set of Lebesgue measure zero. Finally any subset of a set of Lebesgue measure zero is a set of Lebesgue measure zero.

There exist also uncountable sets having Lebesgue measure zero. A typical example of such a set is the Cantor's set. To construct Cantor's set take the closed interval $[0, 1]$ and divide it into three equal intervals. From the middle remove the open interval $(1/3, 2/3)$. The remaining two closed intervals have total length $2/3$ and they form a closed set F_1. Repeat this process with each of the remaining intervals.

After n-steps the remaining set F_n will consist of the union of 2^n disjoint closed intervals of total length of $(2/3)^n$. The sets F_n are nested and their intersection F will represent a nonempty set of cardinality equal to cardinality of the interval $[0, 1]$. The set F can be covered by a countable number of intervals of total length as small as we please. Notice that a finite cover by intervals we can always augment by a sequence of intervals of the form (a, a), that is by empty sets to get a countable cover.

To prove that Cantor's set is of the same cardinality as the interval $[0, 1]$ consider expansions into infinite fractions at the base 3 of points belonging to F

$$x = 0.d_1, d_2, d_3, \ldots$$

where the digits $d_i \in \{0, 2\}$. Ignore the set of points which have periodic expansions since they represent some rational numbers that form a countable set. Clearly all points that do not have periodic expansion are in the Cantor set F.

Similarly consider the binary expansions into nonperiodic sequence of digits of points of the set $[0, 1]$.

$$y = 0.a_1, a_2, a_3, \ldots$$

where $a_i \in \{0,1\}$. Clearly the map $x \mapsto y$ given by the formula

$$a_i = d_i/2 \quad \text{for all} \quad i = 1, 2, \ldots$$

is one-to-one and onto. Thus the cardinalities of F and $[0,1]$ are equal.

It is clear that in the definition of a set $A \subset R$ of Lebesgue measure zero we can restrict ourself just to families consisting of open intervals. Indeed, take any $\varepsilon > 0$ and let the sequence of intervals I_n with end points a_n, b_n be a cover of the set A with the total length of the intervals less than $\varepsilon/2$. Then the sequence

$$J_n = (a_n - 2^{-n-1}\varepsilon, b_n + 2^{-n-1}\varepsilon)$$

consisting of open intervals will cover the set T and its total length

$$\sum\nolimits_{n>0} J_n \leq \sum\nolimits_{n>0} I_n + \varepsilon \sum\nolimits_{n>0} 2^{-n-1}\varepsilon \leq \varepsilon.$$

Thus any set A that can be covered by a sequence of interval of total length as small as we please can be covered by a sequence of open intervals of total length as small as we please. The converse is obvious.

The following theorem characterizes the sets of Lebesgue measure zero.

Theorem 4.2 (Characterization of sets of Lebesgue measure zero)**.** *A set $A \subset R$ is of Lebesgue measure zero if and only if there exists a sequence of open intervals I_n such that*

$$\sum\nolimits_{n=1}^{\infty} |I_n| \leq 1$$

and

$$A \subset \bigcup\nolimits_{n>k} I_n \quad \textit{for all} \quad k = 1, 2, \ldots$$

Proof. Assume that the set A is of Lebesgue measure zero. Thus we can construct for every natural number n a sequence of intervals

$$J_{n,k}\,(k = 1, 2, \ldots)$$

such that

$$A \subset \bigcup\nolimits_{k=1}^{\infty} J_{n,k} \quad \text{and} \quad \sum\nolimits_{k=1}^{\infty} |J_{n,k}| \leq 2^{-n}.$$

Rearrange the double sequence $J_{n,k}$ into a single one I_n and notice that it will have the properties stated in the theorem.

The converse of the above argument follows from the fact that remainder r_n of a convergent series with terms $|I_k|$ converges to zero that is

$$r_n = \sum\nolimits_{k>n} |I_k| \to 0 \quad \text{when} \quad n \to \infty.$$

□

5. A theorem of Riesz

The following theorem can be found in the monograph of F. Riesz and B. Sz.-Nagy [22].

Theorem 5.1 (Riesz)**.** *For every set $T \subset R$ of Lebesgue measure zero there exists a continuous nondecreasing function $g \colon R \mapsto [0,1]$ such that*

$$g'(x) = \infty \quad \textit{for all} \quad x \in T.$$

Proof. For any open interval $I = (a, b)$ define the function $G_I : R \mapsto R$ by the formula

$$\begin{aligned} G_I(x) &= 0 && \text{if} \quad x < a, \\ G_I(x) &= x - a && \text{if} \quad a \le x \le b, \\ G_I(x) &= b - a && \text{if} \quad x > b. \end{aligned} \tag{5.1}$$

For the set T let I_n be a sequence of open intervals as in Theorem 4.2. Let $g_n = G_{I_n}$ for all n. Consider the series with terms g_n. It consists of nonnegative nondecreasing continuous functions. Since

$$g_n(x) \le |I_n| \quad \text{for all} \quad x \in R,\ n = 1, 2, \ldots$$

the series converges uniformly on the entire space R to a continuous function. Thus the function g given by the formula

$$g(x) = \sum\nolimits_{n=1}^{\infty} g_n(x)$$

is well defined and represents a nonnegative nondecreasing and continuous function.

Now let us prove that at every point of the set T the derivative of the function g is equal to infinity. To this end notice that every point $t \in T$ belongs to an infinite number of the intervals I_n. Let k_n denote the number of intervals in the sequence of the first n intervals with index $\le n$,

$$I_1, I_2, \ldots, I_n$$

containing the point t. Let J_n denote the intersection of these intervals. Since t belongs to all of these intervals and the intersection of a finite number of open sets is open, the set J_n is open. So there is a positive number $\delta > 0$ such that

$$(t - \delta, t + \delta) \subset J_n.$$

Now consider the difference quotient

$$\frac{g(x) - g(t)}{x - t}$$

for $|x - t| < \delta$. We have the following representation

$$\frac{g(x) - g(t)}{x - t} = \sum\nolimits_{j \le n} \frac{g_j(x) - g_j(t)}{x - t} + \sum\nolimits_{j > n} \frac{g_j(x) - g_j(t)}{x - t}$$

Since each function g_j is nondecreasing each term of each sum is nonnegative. In the first sum there will be at least k_n terms for which the corresponding difference quotient will be equal to 1. Since the second sum consists of nonnegative terms we have the lower estimate

$$\frac{g(x) - g(t)}{x - t} \ge k_n \quad \text{for all} \quad |x - t| < \delta.$$

Since $k_n \to \infty$ the above estimate proves that $g'(t) = \infty$. □

In the sequel we will need the following lemma.

Lemma 5.2. *Let $I = [a, b]$ denote a closed bounded interval. Let T be a set of Lebesgue measure zero lying in its interior (a, b). There exists a positive, increasing, continuous function $\psi : I \mapsto R$ such that*

$$\psi'(t) = \infty \quad \textit{at every point} \quad t \in T.$$

Proof. Take the function g as in the Riesz theorem (5.1) corresponding to the the set T, and the function G_I as defined in (5.1). Define the function ψ by the formula

$$\psi(x) = g(x) + G_I(x) + 1 \quad \text{for all} \quad x \in [a, b].$$

Clearly this function will satisfy the conditions of the lemma. □

6. Strong Mean-Value theorem

We shall prove the following theorem representing a generalization of a theorem due to Henri Cartan [15], p. 37.

Theorem 6.1 (Strong Mean-Value Theorem)**.** *Let $I = [a, b]$ be a closed bounded interval and let $T \subset (a, b)$ be a set of Lebesgue measure zero. Let $f : I \mapsto Y$ and $g : I \mapsto R$ be continuous functions. Assume that the right-sided Lipschitz derivative $L_+f(x)$ and right-sided lower Dini derivative $D^l_+g(x)$ exist and are finite at every point $x \in (a, b)$. Then the inequality*

$$L_+f(x) \leq D^l_+g(x) \quad \textit{for all} \quad x \in (a, b) \setminus T, \tag{6.1}$$

implies the inequality

$$\|f(b) - f(a)\| \leq g(b) - g(a). \tag{6.2}$$

Proof. Let ψ denote the function from the lemma 5.2 corresponding to the set T.

To prove the theorem it is sufficient to prove the following. For every $\varepsilon > 0$ we have

$$\|f(x) - f(a)\| \leq g(x) - g(a) + \varepsilon((x - a) + \psi(x)) \quad \text{for all} \quad x \in I. \tag{6.3}$$

Once the validity of the statement is established setting $x = b$ and passing to the limit $\varepsilon \to 0$ will yield the inequality (6.2).

We will prove the validity of the above condition by contradiction. Assuming that the statement is not true we get that for some $\varepsilon > 0$ the set

$$U = \{x \in I : \ \|f(x) - f(a)\| > g(x) - g(a) + \varepsilon((x - a) + \psi(x))\} \tag{6.4}$$

is nonempty. Thus from the axiom of continuity follows that the number $c = \inf U$ is well defined. Since all the functions in the inequality (6.3) are continuous, taking all functions from the righthand side of the inequality onto the left side and denoting the function on the left side by ϕ, we get a representation of the set U in the form

$$U = \{x \in I : \ \phi(x) > 0\} = \phi^{-1}(0, \infty),$$

where ϕ is a continuous function. Thus U is an open set in I as the inverse image of an open set by means of a continuous function.

Notice that at $x = a$ the inequality (6.3) is strict since $\psi(0) > 0$. Thus it follows from continuity of the functions involved that for some $d > a$ the inequality (6.3) holds for all $x \in [a, d)$. This means that $U \subset [d, b]$ and thus $a < d \le c$, and so $c \ne a$.

We also have $c \ne b$. Otherwise $U = \{b\}$ and the set U would not be open in I. Hence we must have that $a < c < b$.

The set U cannot contain the point c. Otherwise some closed interval $[c-\delta, c+\delta] \subset (a, b)$, where $\delta > 0$, would be in the set U. Thus $c > c - \delta$, that is the greatest lower bound c of the set U is grater than an element $c-\delta$ of the set U yielding $c > c-\delta \ge c$, a contradiction.

Since $c \notin U$ we must have

$$\|f(c) - f(a)\| \le g(c) - g(a) + \varepsilon((c - a) + \psi(c)). \tag{6.5}$$

Put $\eta = L_+f(c)$ and $\tau = D^l_+g(c)$. From the definitions of lim sup and lim inf we get

$$\eta = \inf_{\delta>0} \sup \left\{ 0 < h < \delta : \ \frac{1}{h}\|f(c + h) - f(c)\| \right\} \tag{6.6}$$

and

$$\tau = \sup_{\delta>0} \inf \left\{ 0 < h < \delta : \ \frac{1}{h}[g(c + h) - g(c)] \right\}. \tag{6.7}$$

From (6.6) follows that for $\eta + \varepsilon/2$ there exists a $\delta_1 > 0$ such that

$$\frac{1}{h}\|f(c + h) - f(c)\| < \eta + \varepsilon/2 \quad \text{if} \quad 0 < h < \delta_1. \tag{6.8}$$

Similarly from (6.7) we get that there exists $\delta_2 > 0$ such that

$$\tau - \varepsilon/2 < \frac{1}{h}[g(c + h) - g(c)] \quad \text{if} \quad 0 < h < \delta_2. \tag{6.9}$$

We have two possibilities that remain either $c \notin T$ or $c \in T$.

First let us consider the case when $c \notin T$. In this case by assumption of the theorem $\eta \le \tau$. The conditions (6.8) and (6.9) yield that for some positive $\delta < \min\{\delta_1, \delta_2\}$ we have

$$\frac{1}{h}\|f(c+h)-f(c)\| \le \eta+\varepsilon/2 \le \tau+\varepsilon/2 \le \frac{1}{h}[g(c+h)-g(c)]+\varepsilon \quad \text{if} \quad 0 < h \le \delta. \tag{6.10}$$

Equivalently

$$\|f(x) - f(c)\| \le [g(x) - g(c)] + \varepsilon(x - c) \quad \text{if} \quad c < x \le c + \delta. \tag{6.11}$$

Thus from the relations (6.5) and (6.11), and the fact that the function ψ is increasing, follows that

$$\begin{aligned} \|f(x) - f(a)\| &\le \|f(x) - f(c)\| + \|f(c) - f(a)\| \\ &\le [g(x) - g(a)] + \varepsilon[(x - a) + \psi(x)] \quad \text{if} \quad c < x \le c + \delta. \end{aligned} \tag{6.12}$$

Thus the number $c + \delta$ is a lower bound of the set U greater then the greatest lower bound $c = \inf U$. A contradiction.

Finally let us consider the case $c \in T$. Since $\psi'(c) = \infty$, we get that for the number $\eta - \tau + \varepsilon$ there exists a positive number δ_3 such that

$$\eta - \tau + \varepsilon < \varepsilon \frac{1}{h}(\psi(c+h) - \psi(c)) \quad \text{if} \quad 0 < |h| < \delta_3. \tag{6.13}$$

Thus we get

$$\eta + \varepsilon/2 < (\tau - \varepsilon/2) + \varepsilon \frac{1}{h}(\psi(c+h) - \psi(c)) \quad \text{if} \quad 0 < |h| < \delta_3. \tag{6.14}$$

Hence for a positive $\delta < \min\{\delta_1, \delta_2, \delta_3\}$ from the relations (6.8) and (6.9) we get

$$\frac{1}{h}\|f(c+h) - f(c)\| \le \frac{1}{h}[g(c+h) - g(c)] + \varepsilon \frac{1}{h}[\psi(c+h) - \psi(c)] \quad \text{if} \quad 0 < h \le \delta \tag{6.15}$$

and therefore

$$\|f(x) - f(c)\| \le [g(x) - g(c)] + \varepsilon[\psi(x) - \psi(c)] \quad \text{if} \quad c < x \le c + \delta. \tag{6.16}$$

Thus from the relations (6.5) and (6.16) we get the relation

$$\begin{aligned} \|f(x) - f(a)\| &\le \|f(x) - f(c)\| + \|f(c) - f(a)\| \\ &\le [g(x) - g(a)] + \varepsilon[(x-a) + \psi(x)] \quad \text{if} \quad c < x \le c + \delta. \end{aligned}$$

and this relation as in the previous case leads to a contradiction. $\square$

7. Henri Cartan's Mean-Value Theorem

Corollary 7.1 (Cartan)**.** *Let $I = [a, b]$ be a closed bounded interval. Let $f : I \mapsto Y$ and $g : I \mapsto R$ be continuous functions. Assume that the right-sided derivatives $f'_r(x)$ and $g'_r(x)$ exist at every point $x \in (a, b)$. Then the inequality*

$$\|f'_r(x)\| \le g'_r(x) \quad \textit{for all} \quad x \in (a, b),$$

implies the inequality

$$\|f(b) - f(a)\| \le g(b) - g(a).$$

Proof. To prove the theorem notice that the existence of right-sided derivatives $f'_r(x)$ and $g'_r(x)$ implies the existence and finiteness of the derivatives $L_+f(x) = \|f'_r(x)\|$ and $D^l_+g(x) = g'_r(x)$. Thus we can use the previous theorem. $\square$

Corollary 7.2. *It is obvious that there exist corresponding theorems for left-sided derivatives. Is is sufficient to introduce functions $\tilde{f}(x) = f(-x)$ and $\tilde{g}(x) = g(-x)$ and observe that right-sided derivatives are mapped onto left-sided derivatives.*

8. A condition for a function to be nondecreasing

Corollary 8.1. *Let $g : [a, b] \mapsto R$ be a continuous function whose right-sided lower Dini derivative $D^l_+ g(x)$ is finite for all x in (a, b). If the derivative is nonnegative at every point of the interval (a, b), with exception perhaps of a set of Lebesgue measure zero, then the function g is nondecreasing on the entire interval $[a, b]$.*

Proof. Define $f(x) = 0$ for all $x \in [a, b]$. Clearly $L_+ f(x) = 0$ on (a, b). Assume $x_1 < x_2$ are any two fixed points of the interval $[a, b]$. If you consider the function g on the interval $[x_1, x_2]$ then all conditions of the Strong Mean-Value Theorem are satisfied on the interval $[x_1, x_2]$. Hence

$$0 = \|f(x_2) - f(x_1)\| \leq g(x_2) - g(x_1) \quad \text{for all} \quad x_1 < x_2.$$

This shows that the function g is nondecreasing. □

9. A sufficient condition for a vector function to be constant

Corollary 9.1 (Sufficient condition for constancy of a vector function)**.** *Let J be an open interval and Y a Banach space. Assume that $f : J \mapsto Y$ is continuous and $f'_r(t) = 0$ for almost all $t \in J$.*

Then the function f is constant on the interval J.

Proof. Introduce function $g(t) = 0$ for all $t \in J$. Take any two different points $t_1, t_2 \in J$. We may assume that $t_1 < t_2$. Let $f(t_1) = y_0$. The pair of functions f, g considered on the interval $[t_1, t_2]$ satisfies the assumptions of the Strong Mean-Value Theorem. Thus we have

$$\|f(t_2) - f(t_1)\| \leq g(t_2) - g(t_1) = 0$$

That is $f(t) = y_0$ for all $t \in J$. □

10. Characterization of Lipschitzian vectorial functions on intervals

Corollary 10.1. *Assume that Y is a Banach space and J any interval. Let $f : J \mapsto Y$ be a continuous function whose right-sided Lipschitz derivative $L_+ f(x)$ is finite for all x in (a, b). If there exists a constant $m < \infty$ such that*

$$L_+ f(x) \leq m$$

at every point of the interval (a, b), with exception perhaps of a set of Lebesgue measure zero, then the function f is Lipschitzian on the entire interval $[a, b]$ and

$$\|f(x) - f(y)\| \leq m|x - y| \quad \textit{for all} \quad x, y \in [a, b].$$

The converse condition is obvious.

Proof. Take any two numbers x, y in the interval $[a, b]$. Without loss of generality we may assume $x > y$. Put $g(x) = mx$ for all $x \in [a, b]$. Consider the closed interval $[y, x]$. It follows from the Strong Mean-Value Theorem that

$$\|f(x) - f(y)\| \leq mx - my = m|x - y| \quad \text{for all} \quad x, y \in [a, b], x > y.$$

□

11. Characterization of Lipschitzian functions on convex sets

Definition 11.1 (Lipschitz derivative). *Now let X, Y denote Banach spaces and W a convex set in X. Consider a function $f : W \mapsto Y$. By a* **Lipschitz derivative** *of f at a point $x \in W$ we shall understand the finite or infinite limit*

$$Lf(x) = \limsup_{y \to x, y \neq x} \frac{\|f(y) - f(x)\|}{\|y - x\|}.$$

Theorem 11.2. *Assume that X and Y are Banach spaces and W is a convex set in X.*

Let $f : W \mapsto Y$ be a continuous function whose Lipschitz derivative $Lf(x)$ is finite for all x in W. If there exists a constant $m < \infty$ such that

$$Lf(x) \leq m$$

at every point of the set W, with exception perhaps of a countable set, then the function f is Lipschitzian on the set W and

$$\|f(x) - f(y)\| \leq m\,\|x - y\| \quad \textit{for all} \quad x, y \in W.$$

The converse is obvious.

12. Right-sided antiderivative of a function

Definition 12.1 (Right-sided antiderivative). *Assume that J is an open interval in R and Y is a Banach space. Assume that f, g are two functions defined on J into the Banach space Y. If the function f is continuous and has right-sided derivative $f'_r(t)$ at every point $t \in J$ and for some set T of Lebesgue measure zero we have the equality*

$$f'_r(t) = g(t) \quad \textit{for all} \quad t \in J \setminus T,$$

then we shall say the the function f forms a **right-sided antiderivative** *of the function g over the interval J.*

Notice that any continuous piecewise linear function or more generally any continuous piecewise differentiable function forms a right-sided antiderivative of its derivative in the above sense. So it is an essential generalization of the notion of antiderivative.

This generalization has another important property stated in the following theorem.

Theorem 12.2 (Any two right-sided antiderivatives of the same function differ by a constat)**.** *Assume that J is an open interval and Y a Banach space. Assume that f_1, f_2, g are defined on the interval J into the Banach space Y. If both f_1 and f_2 form right-sided derivatives of the function g over the interval J then there exists a vector $y_0 \in Y$ such that*

$$f_1(t) - f_2(t) = y_0 \quad \textit{for all} \quad t \in J.$$

Proof. Put $f = f_1 - f_2$. The function f is continuous and from the linearity of the limit operation we get that

$$f_r'(t) = D_+ f(t) = D_+ f_1(t) - D_+ f_2(t) \quad \text{for all} \quad t \in J$$

Moreover

$$f_r'(t) = 0 \quad \text{for almost all} \quad t \in J.$$

Using the Corollary (9.1) we can conclude that for some vector $y_0 \in Y$ we have

$$f_1(t) - f_2(t) = y_0 \quad \text{for all} \quad t \in J.$$

□

Now a natural question arises when we can use the formula known as the Fundamental Theorem of Calculus

$$\int_{t_1}^{t_2} f_r'(u)\, du = f(t_2) - f(t_1) \quad \text{for all} \quad t_1, t_2 \in J.$$

We shall prove later that if the function f_r' is locally bounded on the set J, excluding perhaps a set of Lebesgue measure zero, then the function f_r' is Bochner summable on every closed interval $[t_1, t_2]$ and the the formula (12) holds.

13. Vectorial Lebesgue-Bochner Integration Theory

In this section we shall present a development of the theory of Lebesgue and Bochner spaces of summable functions and present a construction and fundamental theorems of the theory. We shall follow the approach of Bogdanowicz [3] and [4] with some modifications. In the process we shall construct a generalized Lebesgue-Bochner-Stieltjes integral as developed [3].

The development of the integration theory beyond the classical Riemann integral is essential in the modern theory of differential equations, theory of operators, probability, and optimal control theory, and most important in theoretical physics.

Assume that Y, Z, W represent some Banach spaces either over the field R of reals or over the field C of complex numbers.

Denote by U the space of all bilinear bounded operators u from the space $Y \times Z$ into W. Norms of elements in the spaces Y, Z, W, U will be denoted by $|\ |$.

If V is any nonempty collection of subsets of an abstract space X denote by $S(V)$ the family of all sets that are disjoint unions of finite collections of sets from the collection V. Since the empty collection is finite, we implicitly assume that the empty set $\emptyset$ belongs to $S(V)$. The family $S(V)$ will be called the **family of simple sets** generated by the family V.

A nonempty family of sets V of the space X is called a **prering** if the following conditions are satisfied: if $A_1, A_2 \in V$, then both the intersection $A_1 \cap A_2$ and set difference $A_1 \setminus A_2$ belong to the family $S(V)$.

A family of sets V of the space X is called a **ring** if V is a prering such that $V = S(V)$ which is equivalent to the following conditions: if $A_1, A_2 \in V$, then $A_1 \cup A_2 \in V$ and $A_1 \setminus A_2 \in V$. It is easy to prove that a family V forms a prering if and only if the family $S(V)$ of the simple sets forms a ring. Every ring (prering) V of a space X containing the space X itself is called an **algebra (pre-algebra)**, respectively.

If the ring V is closed under countable unions it is called a **sigma ring** (σ-ring for short.) If the ring V is closed under countable intersections it is called a **delta ring** (δ-ring for short.) It follows from de Morgan law that δ-algebra and σ-algebra represent the same notion.

A finite-valued function v from a prering V into $\langle 0, \infty)$, the non-negative reals, satisfying the following implication

$$A = \bigcup_{t \in T} A_t \Longrightarrow v(A) = \sum_{t \in T} v(A_t) \tag{13.1}$$

for every set $A \in V$, that can be decomposed into countable collection $A_t \in V$ ($t \in T$) of disjoint sets, will be called a **σ-additive positive measure.** It was called a **positive volume** in the preceding papers of the author. Notice that since by definition every prering V contains at least one element $A \in V$, we must have that $\emptyset = A \setminus A \in V$. Thus from countable additivity (13.1) follows that $v(\emptyset) = 0$.

By **Lebesgue measure** over an abstract space X we shall understand any set function v from a σ-ring V of the space X into the extended non-negative reals $\langle 0, \infty\rangle$, that satisfies the implication (13.1) and has value zero on the empty set $v(\emptyset) = 0$. We have to postulate this explicitly to avoid the case of a measure that is identically equal to ∞.

As in Halmos [19] a triple (X, V, v), where X denotes an abstract space and V a prering of the space X and v a σ-additive non-negative finite-valued measure on the prering V, will be called a **measure space.**

Halmos considered such measure spaces for the case when V forms a ring of sets. Since every ring satisfies the axioms of a prering, our notion of a measure space is more general.

Halmos used such measure spaces to construct Lebesgue measures and to base on them the development of the integration theory. We reverse the process by first developing the integration theory and obtain the Lebesgue measure as a by product.

It is clear that every finite Lebesgue measure forms a positive measure in our sense, and in the case when it has infinite values by striping it of infinities we obtain a positive measure.

The development of the classical Lebesgue-Bochner theory of the integral goes through the following main stages as in Halmos [19] and Dunford and Schwartz [17]:

- The construction and development of the Caratheodory theory of outer measure v^* over an abstract space X.
- The construction of the Lebesgue measure v on the sigma ring V of measurable sets of the space X induced by the outer measure v^*.
- The development of the theory of real-valued measurable functions $M(v, R)$.

- The construction of the Lebesgue integral $\int f\,dv$.
- The construction and development of the theory of the space $L(v, R)$ of Lebesgue summable functions.
- The construction and development of the theory of the space $M(v, Y)$ of Bochner measurable functions.
- The construction of the Bochner integral $\int f\,dv$ and of the space $L(v, Y)$ of Bochner summable functions f from the space X into any Banach space Y.

The construction of the Lebesgue's integral is an abstraction from the area under the graph of the function similar to the ideas of Riemann though different in execution.

From the point of view of Functional Analysis both the Lebesgue and Bochner integrals are particular linear continuous operators from the space $L(v, Y)$ of summable functions into the Banach space Y. Moreover from the theory of the space $L(v, Y)$ one can easily derive the theory of the spaces $M(v, Y)$, $M(v, R)$, and $L(v, R)$ and of the Lebesgue and Bochner integrals and also the theory of Lebesgue measure. For details see Bogdanowicz [4] and [7].

We shall show in brief how one can develop the theory of the space $L(v, Y)$ and to construct an integral of the form $\int u(f, d\mu)$, where u is any bilinear operator from the product $Y \times Z$ of Banach spaces into a Banach space W and μ represents a vector measure. This integral for the case, when the spaces Y, Z, W are equal to the space R of reals and the bilinear operator u represents multiplication $u(y, z) = yz$, coincides with the Lebesgue integral

$$\int f\,dv = \int u(f, dv) \quad \text{for all} \quad f \in L(v, R). \tag{13.2}$$

In the case, when $Y = W$ and $Z = R$ and $u(y, z) = zy$ represents the scalar multiplication, the integral coincides with the Bochner integral

$$\int f\,dv = \int u(f, dv) \quad \text{for all} \quad f \in L(v, Y). \tag{13.3}$$

It is good to have a few examples of the measure spaces. The first example corresponds to Dirac's δ function.

Example 1 (**Dirac measure space**). *Let X be any abstract set and V the family of all subsets of the space X. Let x_0 be a fixed point of X. Let $v_{x_0}(A) = 1$ if $x_0 \in A$ and $v_{x_0}(A) = 0$ otherwise. Since V forms a sigma ring the triple (X, V, v) forms in this case a Lebesgue measure space.*

Example 2 (**Counting measure space**). *Let X be any abstract set and $V = \{\emptyset, \{x\} :\ x \in X\}$. Let $v(A) = 1$ for all singleton sets $A = \{x\}$ and $v(\emptyset) = 0$. The triple (X, V, v) forms a measure space that is not a Lebesgue measure space.*

Example 3 (**Striped Lebesgue measure space**). *Assume that M is a sigma-ring of subsets of X and μ is any Lebesgue measure on M. Let*

$$V = \{A \in M :\ \mu(A) < \infty\}.$$

Plainly V forms a ring and thus a prering. Then the restriction μ to V yields a measure space $(X, V, \mu.)$

The most important measure space to the sequel is the following.

Proposition 13.1 (Riemann measure space). *Let R denote the space of reals and V the collection of all bounded intervals I open, closed, or half-open. If $a \leq b$ are the end points of an interval I let $v(I) = b - a$. Then the triple (R, V, v) forms a measure space. We shall call this space the* **Riemann measure space.**

Proof. The collection V of intervals forms a prering. Indeed the intersection of any two intervals is an interval or an empty set. But empty set can be represented as an open interval $(a, a) = \emptyset$. The set difference of two intervals is either the union of two disjoint intervals or a single interval or an empty set. Thus we have that for any two intervals I_1, $I_2 \in V$ we have $I_1 \cap I_2 \in S(V)$ and $I_1 \setminus I_2 \in S(V)$. This proves that V is a prering.

To prove countable additivity assume that we have a decomposition of an interval I with ends $a \leq b$ into disjoint countable collection $I_t (t \in T)$ of intervals with end points $a_t \leq b_t$, that is

$$I = \bigcup_{t \in T} I_t. \tag{13.4}$$

The case when interval I is empty or consists of a single point is obvious. So without loss of generality we may assume that the interval I has a positive length and that our index set $T = \{1, 2, 3, \ldots\}$. Take any $\varepsilon > 0$ such that $2\varepsilon < v(I)$. Let $I^\varepsilon = [a + \varepsilon, b - \varepsilon]$ and $I_t^\varepsilon = (a_t - \varepsilon 2^{-t}, b_t + \varepsilon 2^{-t})$ for all $t \in T$.

The family $I_t^\varepsilon (t \in T)$ forms an open cover of the compact interval I^ε thus there exists a finite set $J \subset T$ of indexes such that

$$I^\varepsilon \subset \bigcup_{t \in J} I_t^\varepsilon.$$

The above implies

$$\begin{aligned} v(I) - 2\varepsilon = v(I^\varepsilon) &\leq \sum_{t \in J} v(I_t^\varepsilon) \leq \sum_{t \in T} v(I_t^\varepsilon) \\ &= \sum_{t \in T} (v(I_t) + 2^{-t+1}\varepsilon) = \sum_{t \in T} v(I_t) + 2\varepsilon. \end{aligned}$$

Passing to the limit in the above inequality when $\varepsilon \to 0$ we get

$$v(I) \leq \sum_{t \in T} v(I_t)$$

On the other hand from the relation (13.4) follows that for any finite set J of indexes we have

$$I \supset \bigcup_{t \in J} I_t \Longrightarrow v(I) \geq \sum_{t \in J} v(I_t).$$

Since $\sup_J \sum_{t \in J} v(I_t) = \sum_{t \in T} v(I_t)$ we get from the above relations that the set function v is countably additive and thus it forms a measure. □

As will follow from the development of this theory the Riemann measure space generates the same space of summable functions and the integral as the classical Lebesgue measure over the reals. It is good to see a few more examples of measures related to this one.

Proposition 13.2 (Stieltjes measure space). *Let R denote the space of reals, and g a nondecreasing function from R into R, and D the set of its discontinuity points of g. Let V denote the collection of all bounded intervals I open, closed, or half-open with end points $a, b \notin D$. If $a \leq b$ are the end points of an interval I let $v(I) = f(b) - f(a)$. Then the triple (R, V, v) forms a measure space.*

Proof. The proof is similar to the preceding one and we leave it to the reader. □

A nondecreasing left-side continuous function F from the extended closed interval $E = \langle -\infty, +\infty \rangle$ such that $F(-\infty) = 0$ and $F(+\infty) = 1$ is called a **probability distribution function.** Any measure space (X, V, v) over a prering V such that $X \in V$ and $v(X) = 1$ is called a **probability measure space.**

Proposition 13.3 (Probability distribution generates probability measure space)**.** *Let F be a probability distribution on the extended reals E. Let V consists of all intervals of the form $\langle a, b)$ or $\langle a, \infty \rangle$, where $a, b \in E$. If $I \in V$ let $v(I)$ denote the increment of the function on the interval I similarly a in the case of Stieltjes measure space.*

Then the triple (E, V, v) forms a probability measure space.

Proof. To prove this proposition notice that the space E can be considered as compact space and the proof can proceed similarly as in the case of the Riemann measure space. □

In the case of topological spaces there are two natural prerings available to construct a measure space: The prering consisting of differences $G_1 \setminus G_2$ of open sets, and the prering consisting of differences $Q_1 \setminus Q_2$ of compact sets.

14. Construction of the elementary integral spaces

Definition 14.1 (Vector measure)**.** *A set function μ from a prering V into a Banach space Z is called a* **vector measure** *if for every finite family of disjoint sets $A_t \in V (t \in T)$ the following implication is true*

$$A = \bigcup\nolimits_T A_t \in V \Longrightarrow \mu(A) = \sum\nolimits_T \mu(A_t). \tag{14.1}$$

Denote by $K(v, Z)$ the space of all vector measures μ from the prering V into the space Z, such that

$$|\mu(A)| \leq mv(A) \quad \textit{for all} \quad A \in V \textit{ and some } m.$$

The least constant m satisfying the above inequality is denoted by $\|\mu\|$. It is easy to see that the pair $(K(v, Z), \|\mu\|)$ forms a Banach space.

Assume that c_A denotes the characteristic function of the set A that is $c_A(x) = 1$ on A and takes value zero elsewhere. Let $S(V, Y)$ denote the space of all functions of the form

$$h = y_1 c_{A_1} + \ldots + y_k c_{A_k}, \text{ where } y_i \in Y, A_i \in V. \tag{14.2}$$

The sets A_i in above formula are supposed to be disjoint. Notice also that we extended the multiplication by scalars by agreement $y\lambda = \lambda y$ for all vectors y and scalars λ. The family $S(V, Y)$ of functions will be called the family of **simple functions** generated by the prering V. For fixed $u \in U$ and $\mu \in K(v, Z)$ define the operator

$$\int u(h, d\mu) = u(y_1, \mu(A_1)) + \ldots + u(y_k, \mu(A_k)).$$

Define also

$$\int h dv = y_1 v(A_1) + \ldots + y_k v(A_k).$$

The operators $\int h\, dv$ and $\int u(h, d\mu)$ are well defined, that is, they do not depend on the representation of the function h in the form (14.2).

Let $|h|$ denote the function defined by the formula $|h|(x) = |h(x)|$ for $x \in X$. We see that if $h \in S(V, Y)$, then $|h| \in S(V, R)$. Therefore the following functional $||h|| = \int |h| d\, v$ is well defined for $h \in S(V, Y)$.

The following development of the theory of Lebesgue and Bochner summable functions and of the integrals are from Bogdanowicz [3].

Lemma 14.2 (Elementary integrals on simple functions)**.** *The following statements describe the basic relations between the notions that we have just introduced.*

(1) *The space $S(V, Y)$ is linear, $||h||$ is a seminorm on it, and $\int h\, dv$ is a linear operator on $S(V, Y)$, and $|\int h\, dv| \leq ||h||$ for all $h \in S(V, Y)$.*
(2) *If $g \in S(V, R)$ and $f \in S(V, Y)$, then $gf \in S(V, Y)$.*
(3) *$\int hdv \geq 0$ if $h \in S(V, R)$ and $h(x) \geq 0$ for all $x \in X$.*
(4) *$\int gdv \geq \int fdv$ if $g, f \in S(V, R)$ and $g(x) \geq f(x)$ for all $x \in X$.*
(5) *The operator $\int u(h, d\mu)$ is trilinear from the product space $U \times S(V, Y) \times K(v, Z)$ into the space W and*

$$\left| \int u(h, d\mu) \right| \leq |u|\, ||h||\, ||\mu|| \quad \textit{for all} \quad u \in U,\ h \in S(V, Y),\ \mu \in K(v, Z). \tag{14.3}$$

Let N be the family of all sets $A \subset X$ such that for every $\varepsilon > 0$ there exists a countable family $A_t \in V(t \in T)$ such that $A \subset \bigcup_T A_t$ and $\sum_T v(A_t) < \varepsilon$. Sets of the family N will be called **null-sets.** This family represents a **sigma-ideal** of sets in the power set $\mathcal{P}(X)$, that is, it has the following properties: if $A \in N$, then $B \cap A \in N$ for any set $B \subset X$, and the union of any countable family of null-sets $A_t \in N(t \in T)$ is also a null-set $\bigcup_T A_t \in N$.

A condition $C(x)$ depending on a parameter $x \in X$ is said to be **satisfied almost everywhere** if there exists a set $A \in N$ such that the condition is satisfied at every point of the set $X \setminus A$.

By a **basic sequence** we shall understand a sequence $s_n \in S(V, Y)$ of functions for which there exists a series with terms $h_n \in S(V, Y)$ and a constant $M > 0$ such that $s_n = h_1 + h_2 + \ldots + h_n$, where $||h_n|| \leq M4^{-n}$ for all $n = 1, 2, \ldots$.

Lemma 14.3 (Riesz-Egoroff property of a basic sequence)**.** *Assume that (X, V, v) is a positive measure space on a prering V and Y is a Banach space. Then the following is true.*

(1) [Riesz] *If $s_n \in S(V, Y)$ is a basic sequence, then there exists a function f from the set X into the Banach space Y and a null-set A such that $s_n(x) \to f(x)$ for all $x \in X \backslash A$.*
(2) [Egoroff] *Moreover, for every $\varepsilon > 0$ and $\eta > 0$, there exists an index k and a countable family of sets $A_t \in V(t \in T)$ such that*
$$A \subset \bigcup_T A_t \quad \textit{and} \quad \sum_T v(A_t) < \eta$$
and for every $n \geq k$
$$|s_n(x) - f(x)| < \varepsilon \quad \textit{if} \quad x \notin \bigcup_T A_t.$$

Lemma 14.4 (Dunford's Lemma)**.** *Assume that (X, V, v) is a positive measure space on a prering V and Y is a Banach space. Then the following is true.*

If $s_n \in S(V, Y)$ is a basic sequence converging almost everywhere to zero 0, then the sequence of seminorms $||s_n||$ converges to zero.

15. The Spaces of Lebesgue and Bochner Summable Functions

Definition 15.1 (Lebesgue and Bochner spaces)**.** *Assume that (X, V, v) is a measure space over a prering V of an abstract space X.*

Let $L(v, Y)$ denote the set of all functions $f : X \mapsto Y$, such that there exists basic sequence $s_n \in S(V, Y)$ that converges almost everywhere to the function f.

The space $L(v, Y)$ is called the space of **Bochner summable** *functions and, for the case when Y is equal to the space R of reals, $L(v, R)$ represents the space of* **Lebesgue summable** *functions.*

Define
$$||f|| = \lim_n ||s_n||, \ \int u(f, d\mu) = \lim_n \int u(s_n, d\mu), \ \int f dv = \lim_n \int s_n dv.$$
Since the difference of two basic sequences is again a basic sequence, therefore it follows from the Elementary Lemma 14.2 and Dunford's Lemma 14.4 that the operators are well defined, that is, their values do not depend on the choice of the particular basic sequence convergent to the function f.

Lemma 15.2 (Density of simple functions in $L(v, Y)$)**.** *Assume that (X, V, v) is a positive measure space on a prering V and Y is a Banach space. Let $s_n \in S(V, Y)$ be a basic sequence convergent almost everywhere to a function f. Then $||s_n - f|| \to 0$.*

Theorem 15.3 (Basic properties of the space $L(v, Y)$)**.** *Assume that (X, V, v) is a positive measure space on a prering V and Y is a Banach space. Then the following is true.*

(1) *The space $L(v, Y)$ is linear and $||f||$ represents a seminorm being an extension of the seminorm from the space $S(V, Y)$ of simple functions.*
(2) *We have $||f|| = 0$ if and only if $f(x) = 0$ almost everywhere.*

(3) *The functional* $||f||$ *is a complete seminorm on* $L(v,Y)$ *that is given a sequence of functions* $f_n \in L(v,Y)$ *such that* $\|f_n - f_m\| \overset{nm}{\to} 0$ *there exists a function* $f \in L(v,Y)$ *such that* $\|f_n - f\| \overset{n}{\to} 0$.

(4) *If* $f_1(x) = f_2(x)$ *almost everywhere and* $f_2 \in L(v,Y)$*, then* $f_1 \in L(v,Y)$ *and*

$$||f_1|| = ||f_2||, \ \int f_1 dv = \int f_2 dv, \ \int u(f_1, d\mu) = \int u(f_2, d\mu).$$

(5) *The operator* $\int f dv$ *is linear and represents an extension onto* $L(v,Y)$ *of the operator from* $S(V,Y)$*. It satisfies the condition* $|\int f dv| \leq ||f||$ *for all* $f \in L(v,Y)$.

(6) *The operator* $\int u(f, d\mu)$ *is trilinear on* $U \times L(v,Y) \times K(v,Z)$ *and represents an extension of the operator from the space* $U \times S(V,Y) \times K(v,Z)$*. It satisfies the condition:*

$$|\int u(f, d\mu)| \leq |u| \, ||f|| \, ||\mu|| \quad \textit{for all} \quad u \in U, f \in L(v,Y), \mu \in K(v,Z).$$

From Theorem 15.3 we see that the obtained integrals are continuous under the convergence with respect to the seminorm $\| \ \|$, that if $||f_n - f|| \to 0$, then

$$\int f_n dv \to \int f dv \text{ and } \int u(f_n, d\mu) \to \int u(f, d\mu).$$

The following theorem characterizes convergence with respect to this seminorm.

Theorem 15.4 (Characterization of the seminorm convergence)**.** *Assume that* (X,V,v) *is a positive measure space on a prering* V *and* Y *is a Banach space.*

Assume that we have a sequence of summable functions $f_n \in L(v,Y)$ *and some function* f *from the set* X *into the Banach space* Y*. Then the following conditions are equivalent*

(1) *The sequence* f_n *is Cauchy, that is* $||f_n - f_m|| \overset{nm}{\to} 0$*, and there exists a subsequence* f_{k_n} *convergent almost everywhere to the function* f.

(2) *The function* f *belongs to the space* $L(v,Y)$ *and* $||f_n - f|| \to 0$.

When the space $Y = R$, the space $L(v,R)$ represents the space of Lebesgue summable functions. We have the following relation between Bochner summable functions and Lebesgue summable functions.

Theorem 15.5 (Norm of Bochner summable function is Lebesgue summable)**.** *Let* (X,V,v) *be a positive measure space on a prering* V *and assume that* Y *is a Banach space.*

If f *belongs the the space* $L(v,Y)$ *of Bochner summable functions, then the function* $|f| = |f(\cdot)|$ *belongs to the space* $L(v,R)$ *of Lebesgue summable functions and we have the identity*

$$||f|| = \int |f(\cdot)| \, dv \quad \textit{for all} \quad f \in L(v,Y).$$

Theorem 15.6 (Properties of Lebesgue summable functions)**.** *Let* (X,V,v) *be a positive measure space on a prering* V *and* $L(v,R)$ *the Lebesgue space of* v*-summable functions.*

(a): *If* $f \in L(v,R)$ *and* $f(x) \geq 0$ *almost everywhere on* X *then* $\int f dv \geq 0$.

(b): *If* $f, g \in L(v,R)$ *and* $f(x) \geq g(x)$ *almost everywhere on* X *then* $\int f \, dv \geq \int g \, dv$.

(c): *If* $f, g \in L(v,R)$ *and* $h(x) = \sup\{f(x), g(x)\}$ *for all* $x \in X$ *then* $h \in L(v,R)$.

- **(d):** *Let $f_n \in L(v,R)$ be a monotone sequence with respect to the relation less or equal almost everywhere. Then there exists a function $f \in L(v,R)$ such that $f_n(x) \to f(x)$ almost everywhere on X and $||f_n - f|| \to 0$ if and only if the sequence of numbers $\int f_n\,dv$ is bounded.*
- **(e):** *Let $g, f_n \in L(v,R)$ and $f_n(x) \le g(x)$ almost everywhere on X for $n = 1,2,\ldots$. Then the function $h(x) = \sup\{f_n(x) :\ n = 1,2,\ldots\}$ is well defined almost everywhere on X and is summable, that is, $h \in L(v,R)$. A function defined almost everywhere is said to be summable if it has a summable extension onto the space X.*

From part (d) of the above theorem we can get the classical theorem due to Beppo Levi [20].

Corollary 15.7 (Beppo Levi's Monotone Convergence Theorem)**.** *Assume that (X,V,v) is a positive measure space on a prering V and $L(v,R)$ the Lebesgue space of v-summable functions.*

Let $f_n \in L(v,R)$ be a monotone sequence with respect to the relation less or equal almost everywhere. Then there exists a function $f \in L(v,R)$ such that $f_n(x) \to f(x)$ almost everywhere on X and

$$\int f_n dv \to \int f\,dv$$

if and only if the sequence of numbers $\int f_n\,dv$ is bounded.

Theorem 15.8 (Lebesgue's Dominated Convergence Theorem)**.** *Let (X,V,v) be a positive measure space on a prering V and Y a Banach space.*

Assume that we are given a sequence $f_n \in L(v,Y)$ of Bochner summable functions that can be majorized by a Lebesgue summable function $g \in L(v,R)$, that is for some null set $A \in N$ we have the estimate

$$|f_n(x)| \le g(x) \quad \text{for all} \quad x \notin A \text{ and } n = 1,2,\ldots$$

Then the condition

$$f_n(x) \to f(x) \quad \text{a.e. on } X$$

implies the relations

$$f \in L(v,Y) \quad \text{and} \quad ||f_n - f|| \to 0$$

and, therefore, also the relations

$$\int f_n\,dv \to \int f\,dv \quad \text{and} \quad \int u(f_n, d\mu) \to \int u(f, d\mu)$$

for every bilinear continuous operator u from the product $Y \times Z$ into the Banach space W and any vector measure $\mu \in K(v,Z)$.

16. Continuous functions on compact sets are summable

Now let (X,V,v) denote one of the following measure spaces: (R,V,v) be the Riemann measure space, or the space R^n with the prering consisting of all the cubes of the form

$$A = J_1 \times \ldots \times J_n,\ J_i = (a_i, b_i],\ a_i \le b_i$$

and Riemann measure

$$v(A) = (b_1 - a_1)\cdots(b_n - a_n),$$

or let X be a topological Hausdorff space, the prering V consists of sets of the form $A = Q_1 \setminus Q_2$ where Q_i are compact sets, and the measure v be any countably additive nonnegative finite-valued function on V.

In the case when X is a locally compact topological group the Haar measure restricted to the prering V provides a nontrivial example of such measure space. For details see Halmos [19], Chapters 10 and 11.

The following is Theorem 8, page 498, of Bogdanowicz [3].

Theorem 16.1 (Summability of continuous functions on compact sets)**.** *Assume that the triple (X, V, v) represents one of the above measure spaces and f a continuous function from a compact set $K \subset X$ into a Banach space Y. Then the function f is Bochner summable on K that is we have $c_K f \in L(v, Y)$ and thus the integral $\int_K f(t)\,dt$ exists.*

Proof. Consider first the case of the space R^n. The set K being compact is bounded. Thus there exists a cube I containing the set K. Divide the cube I into finite number of disjoint cubes of diameter less than $\frac{1}{n}$. Let I_j^n $(j = 1, 2, \ldots, k_n)$ be the cubes such that $K \subset \bigcup_j I_j^n$ and $K \cap I_j^n \neq \emptyset$.

In the case of a topological space X the set $f(K)$, as an image of a compact set by means of a continuous function, is compact. There exist disjoint sets I_j^n $(j = 1, 2, \ldots, k_n)$ of diameter $< \frac{1}{n}$ such that $\bigcup_j I_j^n = f(K)$ and each of the sets I_j^n is the intersection of an open set with a closed set. Therefore $f^{-1}(I_j^n) \in V$.

In any case choose $x_j^n \in K \cap I_j^n$. and let

$$s_n = \sum\nolimits_j f(x_j^n) c_{I_j^n} \quad \text{for all} \quad n = 1, 2, \ldots$$

The sequence s_n consists of simple functions thus $s_n \in L(v, Y)$. It converges everywhere on X to the function $c_K f$ and is dominated everywhere by the simple function $m\, c_I$ or $m\, c_K$, where

$$m = \sup\{|f(x)| : x \in K\}.$$

Hence from the Lebesgue Dominated Convergence Theorem we get $c_K f \in L(v, Y)$. □

The above theorems are from Bogdanowicz [3] in the order as they have been proved in that paper.

17. Summable sets form a delta ring

Assume now again that we have a measure space (X, V, v) on a prering V of subsets of an abstract space X. Following Bogdanowicz [5] and [6] denote by V_c the family of all sets $A \subset X$ whose characteristic function c_A is v-summable that is $c_A \in L(v, R)$. Put $v_c(A) = \int c_A dv$ for all sets $A \in V_c$. From the properties of the Lebesgue summable functions (15.6) we can deduce the following proposition.

Proposition 17.1 (Summable sets form a delta ring)**.** *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X.*

Then the family V_c of summable sets forms a δ-ring and the set function v_c forms a measure. If in addition $X \in V_c$ then V_c forms a σ-algebra.

Proof. First of all notice that from Theorem (15.5) follows that absolute value of a Lebesgue summable function is itself summable

$$|f| \in L(v, R) \quad \text{for all} \quad f \in L(v, R). \tag{17.1}$$

Thus by linearity of the space $L(v, R)$ of Lebesgue summable functions and from the identities

$$(f \vee g)(x) = \sup\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|) \quad \text{and}$$
$$(f \wedge g)(x) = \inf\{f(x), g(x)\} = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|)$$

follows that $f \vee g$ and $f \wedge g$ are in $L(v, R)$ if f, g are in $L(v, R)$. Thus from the identities

$$c_{A\cup B} = c_A \vee c_B \quad \text{and} \quad c_{A\cap B} = c_A \wedge c_B \quad \text{and} \quad c_{A\setminus B} = c_A - c_A \wedge c_B$$

we can conclude that the family V_c of summable sets forms a ring.

Now if $A_n \in V_c$ is a sequence of summable sets and $B_n = \bigcap_{j\le n} A_j$ and $B = \bigcap_{j\ge 1} A_j$, from the Dominated Convergence Theorem (15.8) and from the relations

$$|c_{B_n}(x)| = c_{B_n}(x) \le c_{A_1}(x) \quad \text{and} \quad c_{B_n}(x) \to c_B(x) \quad \text{for all} \quad x \in X$$

we get that $B \in V_c$. Thus the family V_c of summable sets forms a δ-ring.

In the case, when $X \in V$, we get from the de Morgan law and the fact that V_c forms a δ-ring that

$$\bigcup_{n\ge 1} A_n = X \setminus \bigcap_{n\ge 1} (X \setminus A_n) \in V_c.$$

Hence in this case V_c forms a σ-algebra.

To show that the triple (X, V_c, v_c) forms a positive measure space assume that $A \in V_c$ and a sequence of disjoint sets $A_n \in V_c$ forms a decomposition of the set A. So $A = \bigcup_{j\ge 1} A_j$. Let $B_n = \bigcup_{j\le n} A_j$. From the Dominated Convergence Theorem (15.8) and from the relations

$$|c_{B_n}(x)| = c_{B_n}(x) \le c_A(x) \quad \text{and} \quad c_{B_n}(x) \to c_A(x) \quad \text{for all} \quad x \in X$$

and linearity of the integral, we get that

$$\begin{aligned} v_c(A) &= \lim_n v_c(B_n) = \lim_n \int c_{B_n} dv \\ &= \lim_n \sum_{j\le n} \int c_{A_j} dv = \lim_n \sum_{j\le n} v_c(A_j) = \sum_j v_c(A_j). \end{aligned}$$

Thus v_c is countably additive on the delta ring V_c. □

Now let us consider the case when the space Y is the space R of reals and Z any Banach space and the bilinear operator u is the multiplication operator $u(r, z) = rz$.

Proposition 17.2 (Isomorphism and isometry of $K(v, Z)$ and $K(v_c, Z)$)**.** *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X.*

Every vector measure $\mu \in K(v, Z)$ can be extended from the prering V onto the delta ring V_c by the formula

$$\mu_c(A) = \int u(c_A, d\mu) \quad \textit{for all} \quad A \in V_c.$$

This extension establishes isometry and isomorphism between the Banach spaces $K(v, Z)$ and $K(v_c, Z)$.

18. A characterization of summable functions

Proposition 18.1 (A characterization of Bochner summable functions)**.** *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space.*

A function f from X into Y belongs to the space $L(v, Y)$ if and only if there exist a sequence $s_n \in S(V, Y)$ of simple functions and a non-negative summable function $g \in L(v, R)$ such that $s_n(x) \to f(x)$ almost everywhere on X and

$$|s_n(x)| \le g(x) \quad \textit{for all} \quad n = 1, 2, \dots \quad \textit{and} \qquad \textit{almost all} \quad x \in X.$$

Proof. If $f \in L(v, Y)$ then there exists a basic sequence of the form

$$s_n = h_1 + h_2 + \cdots + h_n$$

converging almost everywhere to the function f. Notice that the sequence

$$S_n = |h_1| + |h_2| + \cdots + |h_n|$$

is nondecreasing and is basic. Thus it converges almost everywhere to some summable function $g \in L(v, R)$. Since

$$|s_n(x)| \le S_n(x) \le g(x) \quad \text{for almost all} \quad x \in X,$$

we get the necessity of the condition.

The sufficiency of the condition follows from the Dominated Convergence Theorem. □

Proposition 18.2 (Summability of a product of functions)**.** *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space. Let*

$$u(y) = \frac{1}{|y|}\, y \ \textit{if}\ |y| > 0 \quad \textit{and} \quad u(y) = 0 \ \textit{if}\ |y| = 0.$$

Assume that $f \in L(v, Y)$ and $g \in L(v, R)$. Then the product function $u \circ f \cdot g$ is summable that is $u \circ f \cdot g \in L(v, Y)$, where $u \circ f$ denotes the composition $(u \circ f)(x) = u(f(x))$ for all $x \in X$.

Proof. Take any natural number k and define a function

$$u_k(y) = (k|y| \wedge 1)\frac{1}{|y|} y \quad \text{for all} \quad y \in Y,\ |y| > 0 \quad \text{and} \quad u_k(0) = 0.$$

Notice that the function u_k is continuous and

$$\lim_k u_k(y) = u(y) \quad \text{ and } \quad |u_k(y)| \le 1 \quad \text{for all} \quad y \in Y.$$

Let s_n be a basic sequence converging almost everywhere to f and S_n a basic sequence converging almost everywhere to g. Let $G \in L(v,R)$ be a majorant for the sequence S_n. Then we have that the sequence

$$h_{k\,n} = u_k \circ s_n \cdot S_n \in S(V,Y)$$

consists of simple functions and when $n \to \infty$ it converges almost everywhere to the function

$$h_k = u_k \circ f \cdot g.$$

Since G majorizes the sequence $h_{k\,n}$, from the Dominated Convergence Theorem we get $h_k \in L(v,Y)$ and moreover

$$|h_k(x)| \leq G(x) \quad \text{for almost all} \quad x \in X.$$

Passing to the limit $k \to \infty$ and applying the Dominated Convergence Theorem yields that

$$u \circ f \cdot g \in L(v,Y).$$

□

Definition 18.3 (Summability on sets)**.** *Assume that (X,V,v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space.*

We shall say that a function $f : X \mapsto Y$ is **summable on a set** *$A \subset X$ if the product function $c_A f$ is summable and we shall write*

$$\int_A f\,dv = \int c_A f\,dv.$$

Definition 18.4 (Vector measure of finite variation)**.** *A vector measure μ from a prering V into a Banach space Y is said to be of* **finite variation** *on V if*

$$|\mu|(A) = \sup \sum\nolimits_{t\in T} |\mu(A_t)| < \infty \quad \textit{for all} \quad A \in V$$

where the supremum is taken over all finite disjoint decompositions $A_t \in V$ $(t \in T)$ of the set $A = \bigcup_{t\in T} A_t$. The set function $|\mu|$ is called the **variation** *of the vector measure μ.*

Proposition 18.5 (Sets on which a function is summable form a δ-ring)**.** *Assume that (X,V,v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space and assume that f is an arbitrary function from X into Y. Denote by V_f the family of all sets $A \subset X$ on which the function f is summable.*

If $f : X \mapsto Y$ is an arbitrary function then V_f forms a δ-ring and $\mu(A) = \int_A f\,dv$ forms a σ-additive vector measure of finite variation on V_f.

Proof. Assume that $A, B \in V_f$. Then $c_A f \in L(v,Y)$ and $|c_B f| \in L(v,R)$. From Proposition 18.2 we get

$$c_{A\cap B} f = c_A c_B\, u \circ f \cdot |f| = u \circ (c_A f) \cdot |c_B f| \in L(v,Y).$$

Thus $A \cap B \in V_f$. It follow from linearity of the space $L(v,Y)$ and the identities

$$c_{A\setminus B} = c_A - c_{A\cap B} \quad \text{and} \quad c_{A\cup B} = c_{A\setminus B} + c_{A\cap B} + c_{B\setminus A}$$

that V_f forms a ring.

Now using the Dominated Convergence Theorem we can easily prove that V_f forms a δ-ring and the set function

$$\mu(A) = \int_A f\,dv \quad \text{for all} \quad A \in V_f$$

forms a σ-additive vector measure and

$$|\mu|(A) \leq \int_A |f|\,dv < \infty \quad \text{for all} \quad A \in V_f.$$

$\square$

The family of sets V_f on which a function f is summable may consist only of the empty set. However in the case of a summable function this family is rich as follows from the following corollary.

Corollary 18.6 (The collection of sets on which a summable function is summable forms a σ-algebra)**.** *Assume that (X, V, v) is a measure space on a prering V of subsets of an abstract space X and Y a Banach space.*

If $f \in L(v, Y)$ is a summable function then the family V_f of sets, on which f is summable, forms a σ-algebra containing all summable sets that is we have the inclusion

$$V \subset V_c \subset V_f$$

and the set function $\mu(A) = \int_A f\,dv$ is σ-additive of finite total variation $|\mu|(X) \leq \int |f|\,dv$.

Proof. To prove this corollary notice that similarly as before we can prove that product gf of a summable bounded function $g \in L(v, R)$ with a summable function $f \in L(v, Y)$ is summable $gf \in L(v, Y)$. This implies that $V \subset V_c \subset V_f$. $\square$

For further studies of vector measures we recommend Dunford and Schwartz [17], and for extensive survey of the state of the art in the theory of vector measures see the monograph of Diestel and Uhl [16].

19. Extensions to Lebesgue measures

If V is any nonempty collection of subsets of an abstract space X denote by V^σ the collection of sets that are countable unions of sets from V and denote by V^r the collection

$$V^r = \{A \subset X : \ A \cap B \in V \quad \text{for all} \quad B \in V\}.$$

Now assume that (X, V, v) is a measure space and V_c the δ-ring of summable sets and $v_c(A) = \int c_A dv$. It is easy to prove that V_c^σ forms the smallest σ-ring containing the prering V and the family N of v-null sets. Moreover the set function defined by

$$\mu(A) = \sup\{v_c(B) : \ B \subset A,\, B \in V_c\} \quad \text{for all} \quad A \in V_c^\sigma \tag{19.1}$$

forms a Lebesgue measure on V_c^σ.

A Lebesgue measure is called **complete** if all subsets of sets of measure zero are in the domain of the measure and thus have measure zero. The above Lebesgue

measure μ can be characterized as the smallest extension of the measure v to a complete Lebesgue measure. Hence this measure is unique.

One can prove that

$$V_c^r = \bigcap\nolimits_{f \in L(v,Y)} V_f.$$

The family V_c^r as an intersection of σ-algebras forms itself a σ-algebra containing the δ-ring V_c. The smallest σ-algebra V^a containing V_c is given by the formula

$$V^a = \{A \subset X : \ A \in V_c^\sigma \text{ or } X \setminus A \in V_c^\sigma\}.$$

If $X \in V_c^\sigma$ then the sigma algebras coincide $V_c^\sigma = V^a = V_c^r$. If $X \notin V_c^\sigma$, one can always extend the measure v to a Lebesgue measure on V^a or V_c^r by the formula

$$\mu(A) = v_c(A) \text{ if } A \in V_c \quad \text{and} \quad \mu(A) = \infty \text{ if } A \notin V_c. \tag{19.2}$$

However if $\sup\{v_c(A) : \ A \in V_c\} = a < \infty$ and $X \notin V_c^\sigma$ the extensions are not unique. Indeed one can take in this case $\mu(X) = b$, where b is any number from the interval $\langle a, \infty)$, and put

$$\mu(A) = v_c(A) \text{ if } A \in V_c \quad \text{and} \quad \mu(A) = b - v_c(X \setminus A) \text{ if } X \setminus A \in V_c$$

to extend the measure v_c onto the σ-algebra V^a preserving sigma additivity.

Consider an example. Let (X, V, v) be the following measure space:

$$X = R, \quad V = \{\emptyset, \{n\} : \ n = 1, 2, \ldots\}, \quad v(\emptyset) = 0,\, v(\{n\}) = 2^{-n}.$$

In this case the family N of null sets contains only the empty set $\emptyset$, the family S of simple sets consists of finite subsets of the set of natural numbers $\mathcal{N}$, the family V_c of summable sets consists of all subsets of $\mathcal{N}$, we have $V_c^\sigma = V_c$, the smallest σ-algebra extending V_c consists of sets that either are subsets of $\mathcal{N}$ or their complements are subsets of $\mathcal{N}$, finally $V_c^r = P(R)$ consists of all subsets of R. Since $v_c(\mathcal{N}) = 1$ is the supremum of v_c, the measure v_c has many extensions onto the σ-algebras V^a and V_c^σ. One extension onto $P(R)$ is given by 19.2 and another, for instance, by

$$\mu(A) = \sum\nolimits_{n \in A \cap \mathcal{N}} 2^{-n} \quad \text{for all} \quad A \subset R.$$

In view of the existence of a variety of extensions of a measure from a prering onto δ-rings and the multiplicity of extensions to Lebesgue measures it is important to be able to identify measures that generate the same class of Lebesgue-Bochner summable functions $L(v, Y)$ and the same trilinear integral $\int u(f,\, d\mu)$ and thus the ordinary Bochner integral $\int f\, dv$. In this regard we have the following theorems.

Assume that (X, V_j, v_j), $(j = 1, 2)$ are two measure spaces over the same abstract space X and Y, Z, W are any Banach spaces and U is the Banach space of bilinear bounded operators from the product $Y \times Z$ into W.

Theorem 19.1 (When $L(v_2, Y)$ extends $L(v_1, Y)$?)**.** *For every Banach space Y we have $L(v_1, Y) \subset L(v_2, Y)$ and*

$$\int f\, dv_1 = \int f\, dv_2 \quad \text{for all} \quad f \in L(v_1, Y)$$

if and only if $V_{1c} \subset V_{2c}$ and

$$v_{1c}(A) = v_{2c}(A) \quad \text{for all} \quad A \in V_{1c}$$

that is the measure v_{2c} represents an extension of the measure v_{1c}.

Consequently we have the following theorem.

Theorem 19.2. *For any Banach space Y and any bilinear bounded transformation $u \in U$ we have $L(v_1, Y) = L(v_2, Y)$ and*

$$\int f\, dv_1 = \int f\, dv_2 \quad \text{for all} \quad f \in L(v_1, Y)$$

and the spaces of vector measures $K(v_1, Z), K(v_2, Z), K(v_{1c}, Z), K(v_{2c}, Z)$ are isometric and isomorphic and

$$\int u(f, d\mu_1) = \int u(f, d\mu_2) = \int u(f, d\mu_{1c}) = \int u(f, d\mu_{2c}) \quad \text{for all} \quad f \in L(v_1, Y),$$

where $\mu_1, \mu_2, \mu_{1c}, \mu_{2c}$ are vector measures that correspond to each other through the isomorphism, if and only if, the completions of the measures v_1, v_2 coincide $v_{1c} = v_{2c}$.

For proofs of the above theorems see Bogdanowicz [6]. It is important to relate the above theorems to the classical spaces of Lebesgue and Bochner summable functions and the integrals generated by Lebesgue measures. Since there is a great variety of approaches to construct these spaces we shall understand by a classical construction of the Lebesgue space $L(\mu, R)$ the construction developed in Halmos [19] and by classical approach to the theory of the space $L(\mu, Y)$ of Bochner summable functions as presented in Dunford and Schwartz [17].

Now if (X, V, v) is a measure space on a prering V and (X, M, μ) represents a Lebesgue measure space where μ is the smallest extension of the measure v to a Lebesgue complete measure on the σ-ring M, then we have the following theorem.

Theorem 19.3. *For every Banach space Y the spaces $L(v, Y)$ and $L(\mu, Y)$ coincide and we have*

$$\int_A f\, dv = \int_A f\, d\mu \quad \text{for all} \quad f \in L(\mu, Y) \quad \text{and} \quad A \in M.$$

The above theorem is a consequence of the theorems developed in Bogdanowicz [4].

20. Tensor product of measure spaces

Assume now that we have two measure spaces (X_i, V_i, v_i) over abstract spaces X_i for $i = 1, 2$. Consider the Cartesian product $X_1 \times X_2$. By **tensor product** $V_1 \otimes V_2$ of the families V_i we shall understand the family of sets

$$V_1 \otimes V_2 = \{A_1 \times A_2 : \; A_1 \in V_1 \text{ and } A_2 \in V_2\}.$$

We shall use a shorthand notation

$$v_1 \otimes v_2(A_1 \times A_2) = v_1(A_1)v_2(A_2) \quad \text{for all} \quad A_i \in V_i \, (i = 1, 2).$$

Theorem 20.1 (Tensor product of measure spaces)**.** *Assume that (X_i, V_i, v_i) are measures over abstract spaces X_i and V_i are prerings for $i = 1, 2$. Let the triple (X, V, v) consist of $X = X_1 \times X_2$, $V = V_1 \otimes V_2$, and $v(A) = v_1(A_1)v_2(A_2)$ for all $A = A_1 \times A_2 \in V$. Then V forms a prering and v a σ-additive finite-valued positive measure, that is the triple (X, V, v) forms a measure space.*

Proof. Notice that the following two properties of a family V of subsets of a space X are equivalent:

- The family V forms a prering.
- The empty set belongs to V and for every two sets $A, B \in V$ there exists a finite disjoint refinement from the family V, that is, there exists a finite collection $\{D_1, \ldots, D_k\}$ of disjoint sets from V such that each of the two sets A, B can be represented as a union of some sets from the collection.

Clearly the tensor product $V_1 \otimes V_2$ of the prerings contains the empty set. Now take any pair of sets $A, B \in V_1 \otimes V_2$. We have $A = A_1 \times A_2$ and $B = B_1 \times B_2$. If one of the sets A_1, A_2, B_1, B_2 is empty then the pair A, B forms its own refinement from V. So consider the case when all the sets A_1, A_2, B_1, B_2 are nonempty.

Let $C = \{C_j \in V_1 : \ j \in J\}$ be a refinement of the pair A_1, B_1 and

$$D = \{D_k \in V_2 : \ k \in K\}$$

a refinement of A_2, B_2. We may assume that the refinements do not contain the empty set.

The collection of sets $C \otimes D$ forms a refinement of the pair A, B. Indeed each set of the pair A_1, B_1 can be uniquely represented as the union of sets from the refinement C. Similarly each set of the pair A_2, B_2 can be represented in a unique way as union of sets from the refinement D. Since the sets of the collection $C \otimes D$ are disjoint and nonempty, each set of the pair $A_1 \times A_2$ and $B_1 \times B_2$ can be uniquely represented as the union of the sets from $C \otimes D$. Thus $V = V_1 \otimes V_2$ is a prering.

To prove that the set function $v = v_1 \otimes v_2$ is σ-additive take any set $A \times B$ in V and let $A_n \times B_n \in V$ denote a sequence of disjoint sets whose union is the set $A \times B$. Notice the identity

$$c_A(x_1)\, c_B(x_2) = \sum\nolimits_n c_{A_n}(x_1)\, c_{B_n}(x_2) \quad \text{for all} \quad x_1 \in X_1, x_2 \in X_2. \tag{20.1}$$

Fixing x_2 and integrating with respect to v_1 both sides of the equation (20.1) on the basis of the Monotone Convergence Theorem, we get

$$v_1(A)\, c_B(x_2) = \sum\nolimits_n v_1(A_n)\, c_{B_n}(x_2) \quad \text{for all} \quad x_2 \in X_2.$$

Integrating the above term by term with respect to v_2 and applying again the Monotone Convergence Theorem yields

$$v_1(A)\, v_2(B) = \sum\nolimits_n v_1(A_n)\, v_2(B_n)$$

that is the set function

$$v(A \times B) = v_1 \otimes v_2(A \times B) = v_1(A) v_2(B) \quad \text{for all} \quad A \times B \in V_1 \otimes V_2$$

is σ-additive. Hence the triple

$$(X, V, v) = (X_1 \times X_2, V_1 \otimes V_2, v_1 \otimes v_2)$$

forms a measure space. ☐

The above theorem has an immediate generalization to any finite number of measure spaces.

Theorem 20.2 (Tensor product of n measure spaces)**.** *Assume that (X_i, V_i, v_i) are measures over abstract spaces X_i and V_i are prerings for $i = 1, \ldots, n$. Let the triple (X, V, v) consist of $X = X_1 \times \cdots \times X_n$, $V = V_1 \otimes \cdots \otimes V_n$, and*

$$v(A) = v_1 \otimes \cdots \otimes v_n(A) = v_1(A_1) \cdots v_n(A_n)$$

for all $A = A_1 \times \cdots \times A_n \in V$. Then the triple (X, V, v) forms a measure space.

Definition 20.3 (Classical Lebesgue measure over R^n)**.** *To construct the classical Lebesgue measure μ over the space R^n, first take the tensor product of n copies of the Riemann measure space (R, V, v), and complete it by means of the formula (19.1).*

21. Integration over the space R of reals

Now let $X = I$ be a closed bounded interval, and let V denote the prering of all subintervals of I, and $v(A)$ the length of the interval $A \subset I$. Clearly the space (X, V, v) is a measure space as a subspace of the Riemann measure space.

For the case of Riemann measure space we shall use the customary notation for the integral of a Bochner summable function $f \in L(v, Y)$. We shall write

$$\begin{aligned} \int_{t_1}^{t_2} f(t)\,dt &= \int c_{[t_1,t_2]} f\,dv \quad \text{if} \quad t_1 \le t_2,\ t_1, t_2 \in I, \\ \int_{t_1}^{t_2} f(t)\,dt &= -\int c_{[t_2,t_1]} f\,dv \quad \text{if} \quad t_1 > t_2,\ t_1, t_2 \in I. \end{aligned}$$

Adopting the above notation yields a convenient formula for any $f \in L(v, Y)$

$$\int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^{t_3} f(t)\,dt + \int_{t_3}^{t_1} f(t)\,dt = 0 \quad \text{for all} \quad t_1, t_2, t_3 \in I.$$

22. Right-sided antiderivatives and fundamental theorem of calculus

We remind the reader that the notion of a set of Lebesgue measure zero is the same as the notion of the null set corresponding to the Riemann measure space over the reals.

We also use the notion of locally bounded, or locally essentially bounded, or locally summable over an open set J to mean that each point of the set J has a neighborhood on which this property holds. Notice that the notion that a function f is locally bounded, or locally essentially bounded, locally summable on J is equivalent to the property that the function is bounded, or essentially bounded, or summable on every compact subset F of the set J.

Theorem 22.1 (If f'_r is locally essentially bounded then fundamental theorem of calculus holds)**.** *Let J be an open interval and Y a Banach space. Assume that f, g are defined on J into Y. If the function f represents a right-sided antiderivative of g over the interval J and g is locally bounded on $J \setminus T$ where T is a set of Lebesgue measure zero, then g is Bochner summable and we have*

$$\int_{t_1}^{t_2} g(u)\,du = f(t_2) - f(t_1) \quad \textit{for all} \quad t_1, t_2 \in J.$$

Proof. Take any two points $t_1, t_2 \in J$. We may assume without loss of generality that $t_1 < t_2$. Select $\delta > 0$ so that $t_2 + \delta \in J$.

Take any sequence $h_n \in (0, \delta)$ such that $h_n \to 0$ and consider the functions

$$g_n(t) = (h_n)^{-1}[f(t + h_n) - f(t)] \quad \text{for all} \quad t \in I = [t_1, t_2].$$

Notice that the functions g_n are continuous on the interval I and since I is compact the functions g_n are summable on I.

Since neglecting a set T of Lebesgue measure zero the function g is local bounded on J, from compactness of the interval I we get that there is a constant m such that

$$\|g(t)\| \le m \quad \text{for all} \quad t \in [t_1, t_2 + \delta] \setminus T.$$

Thus from the Strong Mean-Value theorem (6.2 follows that

$$\|g_n(t)\| \le m \quad \text{for all} \quad t \in [t_1, t_2 + \delta] \setminus T.$$

Since by definition of right-sided antiderivative

$$g_n(t) \to f'_r(t) \quad \text{for all} \quad t \in I \quad \text{ and } \quad f'_r(t) = g(t) \quad \text{for all} \quad t \in I \setminus T$$

from the Dominated Convergence theorem (15.8) we get that the function g is Bochner summable on the interval I and we have the convergence

$$\int_{t_1}^{t_2} g_n(u)\,du \to \int_{t_1}^{t_2} g(u)\,du.$$

Now from the fact that the Riemann measure is invariant under translation follows that also the integral is invariant under translation. Using this fact and continuity of the function f we get

$$\begin{aligned}\int_{t_1}^{t_2} g_n(u)\,du &= (h_n)^{-1}\left[\int_{t_1}^{t_2} f(u + h_n)\,du - \int_{t_1}^{t_2} f(u)\,du\right] \\ &= (h_n)^{-1}\left[\int_{t_1+h_n}^{t_2+h_n} f(u)\,du - \int_{t_1}^{t_2} f(u)\,du\right] \\ &= (h_n)^{-1}\int_{t_2}^{t_2+h_n} f(u)\,du - (h_n)^{-1}\int_{t_1}^{t_1+h_n} f(u)\,du\end{aligned}$$

Hence

$$\int_{t_1}^{t_2} g_n(u)\,du \to f(t_2) - f(t_1)$$

Therefore we must have

$$\int_{t_1}^{t_2} g(u)\,du = f(t_2) - f(t_1) \quad \text{for all} \quad t_1, t_2 \in J.$$

□

Theorem 22.2. *Let J be an open interval and Y a Banach space. Assume that g from J into the Banach space Y is locally summable on J with respect to Riemann measure. If the function g is right-side continuous on the interval J then the function*

$$f(t) = f(t_0) + \int_{t_0}^{t} g(u)\,du \quad \textit{for all} \quad t \in J$$

represents a right-sided antiderivative of g over the interval J.

Proof. The proof is straightforward and we leave it to the reader. □

23. Lebesgue points of a function summable with respect to Riemann measure

A summable function f with respect to the Riemann measure space is said to have a Lebesgue point at $t = s$ if

$$\lim_{h\to 0} \frac{1}{h} \int_{s}^{s+h} |f(t) - f(s)|\,dt = 0.$$

At a Lebesgue point the function $F(t) = \int_{t_0}^{t} f(u)\,du$ is differentiable and $F'(s) = f(s)$.

Theorem 23.1 (Almost all points of a summable function are Lebesgue points)**.** *Assume that (R, V, v) is the Riemann measure space and Y a Banach space. Then almost every point of any summable function $f \in L(v, Y)$ is a Lebesgue point. Therefore for every such a function f the function*

$$F(t) = \int_{t_0}^{t} f(u)\,du \quad \textit{for all} \quad t \in R$$

is differentiable almost everywhere and we have that $F' = f$ almost everywhere on R.

For a proof of this powerful theorem see Dunford and Schwartz [17, page 217, Theorem 8].

Definition 23.2 (Absolute continuity of a vector measure)**.** *Given a Lebesgue measure space (X, V, v) on a σ-algebra V and σ-additive vector measure μ from V into a Banach space Y. We say that the vector measure μ is absolutely continuous with respect to the measure v if*

$$\mu(A) = 0 \quad \textit{whenever} \quad v(A) = 0.$$

Theorem 23.3 (Phillips)**.** *Assume that (X, V, v) is a Lebesgue measure space on a σ-algebra V such that $v(X) < \infty$ and Y is a reflexive Banach space.*

Assume that μ is a σ-additive vector measure of finite variation from the σ-algebra V into Y.

If μ is absolutely continuous with respect to the Lebesgue measure v, then there exists a Bochner summable function $g \in L(v, Y)$ such that

$$\mu(A) = \int_A g\,dv \quad \textit{for all} \quad A \in V.$$

For the proof of this remarkable theorem see Diestel and Uhl [16, page 76]. This result can be found in the original paper of Phillips [21].

Theorem 23.4 (Integral representation of Lipschitzian functions)**.** *Let I denote any closed bounded interval of the space R of reals and H a Hilbert space over the field R. Assume that (I, V, v) is the Riemann measure space as above. Let f denote a function from the interval I into the Hilbert space H.*

If the function f is Lipschitzian on the interval I, that is

$$|f(t_1) - f(t_2)| \leq m\,|t_1 - t_2| \quad \textit{for all} \quad t_1, t_2 \in I,$$

then there exists a Bochner summable function $g \in L(v, H)$ such that $|g(t)| \leq m$ for all $t \in I$ and

$$f(t) - f(t_0) = \int_{t_0}^{t} g(u)\,du \quad \textit{for all} \quad t, t_0 \in I. \tag{23.1}$$

Moreover the function f is differentiable almost everywhere and $f'(t) = g(t)$ for almost all $t \in I$.

Proof. It follows from Riesz representation theorem that every Hilbert space is reflexive. Define a set function μ on an interval $A \subset I$ with end points $a \leq b$ by

$$\mu(A) = f(b) - f(a) \quad \text{for all intervals} \quad A \in V.$$

The function μ is additive on the prering V. Thus it forms a vector measure. Since f is Lipschitzian with constant m we have

$$|\mu(A)| = |f(b) - f(a)| \leq m\,(b - a) = m\,v(A) \quad \text{for all} \quad A \in V,$$

thus the vector measure μ belongs to the space $K(v, H)$ and $\|\mu\| \leq m$. Introduce the bilinear transformation by $u(r, h) = rh$ for all $r \in R$ and $h \in H$. Define the set function μ_c by

$$\mu_c(A) = \int u(c_A, \mu) \quad \text{for all} \quad A \in V_c.$$

Since the interval I belongs to the prering V, the family V_c of summable sets forms a σ-algebra. It follows form linearity of the integral $\int u(f, d\mu)$ with respect to f that the set function μ_c is additive and we have

$$|\mu_c(A)| = |\int u(c_A, d\mu)| \leq |u|\ \|c_A\|\,\|\mu\| = mv_c(A) \quad \text{for all} \quad A \in V_c,$$

The above implies that the vector measure μ_c is σ-additive and absolutely continuous with respect to the Lebesgue measure v_c. Thus by theorem of Phillips 23.3 there exists a summable function $g \in L(v_c, H) = L(v, H)$ such that

$$\mu_c(A) = \int_A g\, dv_c \quad \text{for all} \quad A \in V_c.$$

In particular

$$\mu(A) = \mu_c(A) = \int_A g\, dv_c = \int_A g\, dv \quad \text{for all} \quad A \in V$$

that is

$$f(t_2) - f(t_1) = \int_{\langle t_1, t_2 \rangle} g\, dv = \int_{t_1}^{t_2} g(s)\, ds \quad \text{for all} \quad t_1 \le t_2,\ t_1, t_2 \in I.$$

It follows from Theorem 19.1 that the function $c_I g$ is Bochner integrable with respect to Riemann measure space (R, V, v). Thus almost all points of $c_I g$ are its Lebesgue points. If s lies inside the interval I and s is a Lebesgue point of the function $c_I g$ then for sufficiently small δ

$$|\frac{1}{h}(f(s+h) - f(s))| \le \left| \frac{1}{h} \int_s^{s+h} |g(t) - g(s)|\, dt \right| \le m \quad \text{for all} \quad |h| < \delta.$$

Passing to the limit in the above inequality we get $|f'(s)| = |g(s)| \le m$. Replacing eventually the values of $g(s)$ by zero, on a set of measure zero, we may assume that $|g(t)| \le m$ for all $t \in I$. □

For a direct proof of the above theorem see Bogdanowicz and Kritt [8].

Definition 23.5 (Space $M_\infty(I, Y)$)**.** *Let (R, V, v) be the Riemann measure space and I a closed bounded interval. Assume that Y is a Banach space. Denote by $M_\infty(I, Y)$ the set of all functions that are Bochner summable on the interval I and bounded almost everywhere. Introduce the functional*

$$\|f\| = \inf \{M : \ |f(x)| \le M \quad \textit{for almost all} \quad x \in I\}.$$

The functional $\| \ \|$ forms a seminorm on $M_\infty(I, Y)$. Identify functions that are equal almost everywhere. After such identification the pair $(M_\infty(I, Y), \| \ \|)$ becomes a Banach space. The space obtained in this way is called the **space of essentially bounded measurable functions** *and the norm* **essential sup norm.**

Proposition 23.6 (The space $C(I, Y)$ forms a subspace of $M_\infty(I, Y)$)**.** *The map $f \mapsto [f]$, where $[f]$ denotes the collection of all functions that are equal almost everywhere to the function f, establishes an isometry and isomorphism between the space $C(I, Y)$ and a subspace of the space $M_\infty(I, Y)$.*

Proof. The proof is straightforward and we leave it to the reader. □

References

[1] S. Banach, *Theory of Linear Operations*, Amsterdam: North-Holland, 1987.

[2] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fund. Math. 20, (1933),p. 262–276.

[3] W.M. Bogdanowicz, (also known as V.M. Bogdan), *A Generalization of the Lebesgue-Bochner-Stieltjes Integral and a New Approach to the Theory of Integration*, Proc. of Nat. Acad. Sci. USA, Vol. 53, No. 3, (1965), p. 492–498
(On the web: http://faculty.cua.edu/bogdan/rep/10.pdf).

[4] ——— *An Approach to the Theory of Lebesgue-Bochner Measurable Functions and to the Theory of Measure*, Math. Annalen 164, (1966), p. 251–269.
(On the web: http://faculty.cua.edu/bogdan/rep/14.pdf).

[5] ——— *Existence and uniqueness of extensions of volumes and the operation of completion of a volume.* Proceedings of the Japan Academy 42 (1966):571-576
(On the web: http://faculty.cua.edu/bogdan/rep/15.pdf).

[6] ——— *On volumes generating the same Lebesgue - Bochner integration.* Proceedings of the National Academy of Sciences USA 56 (1966):1399-1405
(On the web: http://faculty.cua.edu/bogdan/rep/20.pdf).

[7] ——— *An approach to the theory of integration and theory of Lebesgue - Bochner measurable functions on locally compact spaces.* Mathematische Annalen 171 (1967):219-238
(On the web: http://faculty.cua.edu/bogdan/rep/23.pdf).

[8] W.M. Bogdanowicz, and Kritt, B., *Radon - Nikodym differentiation of one vector-valued volume with respect to another.* Bulletin de l'Academie Polonaise des Sciences 15 (1967):479-486
(On the web: http://faculty.cua.edu/bogdan/rep/22.pdf).

[9] V.M. Bogdan, *Existence of solutions to differential equations of relativistic mechanics involving Lorentzian time delays,* Journal of Mathematical Analysis and Applications, Academic Press, 118, No. 2, September (1986):561-573
(On the web: http://faculty.cua.edu/bogdan/rep/61.pdf).

[10] V.M. Bogdan, *Feynman's Electromagnetic Fields Induced by Moving Charges and the Existence and Uniqueness of Solutions to N-Body Problem of Electrodynamics,* Quaestiones Mathematicae, vol. 32, (2009):1-87

[11] V.M. Bogdan, *Fields generated by a moving relativistic point mass and mathematical correction to Feynman's law* (to appear in Quaestiones Mathematicae), (Abstract is available on the web
http://arxiv.org/abs/0909.5240)

[12] V.M. Bogdan, *Electromagnetic Field Generated by a Moving Plasma and Feynman and Liénard-Wiechert Formulas for a Moving Point Charge,* presented at annual meeting of the Society for Applied and Industrial Mathematics, Denver, Colorado, 7 July 2009 http://arxiv.org/abs/0910.0274

[13] V.M. Bogdan, *Relativistic gravity fields and electromagnetic fields generated by flows of matter*, (these archives) http://arxiv.org/abs/0910.0538

[14] V.M. Bogdan, *Maxwell Equations, amended Feynman's Law, and the Existence and Uniqueness of Solutions to N-Body Problem of Relativistic Gavitational Electrodynamics,* monograph, (in preparation)

[15] H. Cartan, *Differential Calculus*, Hermann, Paris, 1971.

[16] J. Diestel and J.J. Uhl, *Vector Measures,* American Mathematical Society, Providence, Rhode Island (1977)

[17] N. Dunford and J. T. Schwartz *Linear Operators, Part I: General Theory* Interscience Publishers, New York, N.Y., (1958)

[18] A. Einstein, *Zur Elektrodynamik bewegter Kōrper,* Annalen der Physik. 17, 891: 1905.
Enlish translation: *On the Electrodynamics of Moving Bodies,* The Principle of Relativity, Methuen and Company, 1923 and on the web at
http://www.fourmilab.ch/etexts/einstein/specrel/www/

[19] P.R. Halmos, *Measure Theory,* Van Nostrand, New York, N.Y. (1950)

[20] B. Levi, *Sopra l'integratione delle serie,* Rend. Instituto Lombardo di Sci. et Lett., (2) vol. 39 (1906), p. 775-780

[21] R.S. Phillips, *On weakly compact subsets of a Banach space,* Amer. J. Math. 65, (1943), p. 108-136
[22] F. Riesz, and B. Sz-Nagy, *Functional Analysis,* Dover Publishing, Mineola, N.Y., 1990

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